Dynamic Stable Matching

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1 Introduction

The Stable Matching (SM) problem is a classical problem in computational social choice that attempts to find a stable match between two equally sized opposing sets of agents, where each agent reports a vector of preferences over agents in the opposing set. In the traditional setting, preferences or rankings are static and the fundamental goal is to find a perfect, stable match: a match where each unique man and woman are paired such that no pair of man m or woman w prefer each other to their current match. In this work, we aim to investigate a variant of this stable matching problem, where agents' rankings are dynamic over time. Specifically, we investigate the more realistic scenario where agents have dynamic utility vectors over the set of opposing agents. Thus, these utility vectors may change at each time step, thus changing the underlying ranking of each agent over time.

Under this dynamic setting, we will investigate the use of various algorithms to analyze how consistency of matchings and social welfare change over time. Specifically, we investigate the following:

- Men Preferred Deferred Acceptance (MPDA) under different dynamics of utilities. We will analyze how social welfare and consistency of matches varies as the agents' utilities are subject to various dynamics imposed by a Gaussian probability distribution
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2 Related Works

Stable Matching and Variants. In their 1962 paper [2], Gale and Shapeley first proposed the classical stable matching problem and the corresponding Gale-Shapeley algorithm, with two versions: the Men-Proposing Deferred Acceptance (MPDA) algorithm and the Women-Proposing Deferred Acceptance (MPDA) algorithm. Both versions are proven to generate a stable matching and terminate in n^2 moves and less [3]. MPDA is man-optimal where every women gets her worst possible partner, while WPDA is woman-optimal. To address the imbalance, Irving et al. [4] further proposed a sex-optimal stable matching algorithm that operates at $\mathcal{O}(n^4)$.

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There are many variants of the stable matching problem. The survey paper by Iwama and Miyazaki [5] provides a comprehensive overview of the popular variants, including the many-to-one matching (Hospitals/Residents Matching Problem), non-bipartite matching (Roommates Matching Problem), many-to-many matching, three-dimensional matching with three parties, and preferences with incomplete lists and/or ties. In addition, Anshelevich et al. [1] extends the preference profile with numerical utilities to analyze the overall social welfare of stable matching. We inherits Anshelevich's notion of numerical utilities when evaluating social welfare. Nevertheless, none of the existing settings considers the dimension of time, so our setting remains novel.

3 Problem Formulation

In this section, we formally describe the dynamic stable matching setting, followed by several desirable properties that a matching should have.

3.1 Problem Setting

In the classical stable matching problem setting, there are N men and N women where each man m_i has a strict preference ranking $\overrightarrow{\succ}_{m_i}$ for the women and vice versa, and the goal is to find a perfect and stable matching M such that no pair of man m and woman w prefer each other to their current matches. In this project, we study a variant of the stable matching problem with the modification that each agent's preference is induced by underlying numerical utilities, and that the utilities may change over time, prompting a need to re-match the agents for stability.

More formally, in our *dynamic* stable matching setting, there are again N men and N women. Each man m_i is associated with an underlying numerical utility vector $u_i \in \mathbb{R}^n$ satisfying $w_{j_1} \succ_{m_i} w_{j_2} \Leftrightarrow u_i(w_{j_1}) > u_i(w_{j_2})$ for every pair of women (w_{j_1}, w_{j_2}). Similarly, each women w_j is associated with a utility vector $v_j \in \mathbb{R}^n$ consistent with her preferences for the group of men. The utilities are normalized, i.e., $\sum_{j=1}^{N} u_i(w_j) = 1$ for every m_i, and $\sum_{i=1}^{N} v_j(m_i) = 1$ for every w_j.

In addition, each agent also has an excitement vector for each viable candidate. Formally, each man m_i has an excitement vector $x_i \in \mathbb{R}^n$ where each $x_i(w_j)$ denotes the excitement man m_i has for woman w_j. Similarly, each woman w_j has an excitement vector $y_j \in \mathbb{R}^n$ representing her excitement towards each man. All excitement vectors are non-negative.

Furthermore, we add a time dimension to the problem. For simplicity, instead of working with the infinite continuous time domain, we define a finite set of discrete times $t \in \{1, \dots, T\}$, and each person's utility vector will change according to the excitement and and current matching over time according to the following rule. At each time step t, with everyone's utilities $\overrightarrow{u}^t$ and $\overrightarrow{v}^t$ and the computed matching M^t, we derive the updated utilities $\overrightarrow{u}^{t+1}$ and $\overrightarrow{v}^{t+1}$ for each man m_i and woman w_j as follows:

$$u_i^{t+1}(w_j) = \begin{cases} u_i^t(w_j) * (1 + x_i(w_j)) & \text{if } M^t(m_i) \neq w_j \\ u_i^t(w_j) * (1 - x_i(w_j)) & \text{if } M^t(m_i) = w_j \end{cases},$$

$$v_j^{t+1}(m_i) = \begin{cases} v_j^t(m_i) * (1 + y_j(m_i)) & \text{if } M^t(w_j) \neq m_i \\ v_j^t(m_i) * (1 - y_j(m_i)) & \text{if } M^t(w_j) = m_i \end{cases}.$$

In other words, the matched couple have decreased utilities towards each other and increased utilities for all other candidates. As a result, we will need to re-run the matching algorithm to generate a new matching

M^{t+1}. Depending on the updated utilities $\overrightarrow{u}^{t+1}$ and $\overrightarrow{v}^{t+1}$, the couples from M^t may not stay together in the new matching M^{t+1}.

3.2 Metrics

We design this new setting with inspirations from relationships in the real world, where people can divorce and enter new marriages over time due to changed utilities towards each other. We evaluate the matches with the following metrics.

Stability We measure the instability of a series of matches $(M^1, \ldots, M^T)$ with the mean number of blocking pairs, i.e.,

$$\text{instability}(M^1, \ldots, M^T) = \frac{1}{T} \sum_{t=1}^{T} \frac{\sigma(M^t)}{N}, \tag{1}$$

where $\sigma(M^t)$ is the number of blocking pairs in M^t.

Social Welfare We define the social welfare of a series of matches as the mean utilities each agent derives from the matching:

$$\text{sw}(M^1, \ldots, M^T) = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{2N} \sum_{(m_i, w_j) \in M^t} u_i(w_j) + v_j(m_i). \tag{2}$$

Consistency We also would like to encourage the couples to stay together over time. We define the consistency of a series of matches to be:

$$\text{consistency}(M^1, \ldots, M^T) = \frac{1}{T-1} \sum_{t=1}^{T-1} |\{(m, w) : (m, w) \in M^t \wedge (m, w) \in M^{t+1}\}|. \tag{3}$$

Stability vs. Social Welfare Note that stability and social welfare are both metrics that take into account of the preference profiles/utilities of the agents, but one does not necessarily imply the other. It is easy to show that a max social welfare matching M is not necessarily stable, and a stable matching might not maximize social welfare.

Max social welfare $\not\Rightarrow$ stability: Consider two men and two women where m_1 has utility 1 towards w_1 and utility 0 towards w_2, m_2 has utility 0.6 towards w_1 and 0.4 towards w_2, w_1 has utility 0.4 towards m_1 and 0.6 towards m_2, and w_2 has utility 0 towards m_1 and 1 towards m_2. In this setting, $M = \{(w_1, m_1), (w_2, m_2)\}$ is the matching that maximizes social welfare, but it is not stable as m_2 and w_1 both prefer each other more than the matched partner.

Stability $\not\Rightarrow$ max social welfare: In the setting above, $M' = \{(m_1, w_2), (m_2, w_1)\}$ is a stable pair, but it achieves less social welfare than M.

Our goal is to empirically evaluate a few matching algorithms in different settings of utility and excitement initialization, and identify algorithms that can produce matches with high stability, high social welfare and high consistency.

4 Experiments

4.1 Experimental Set-up

Dynamics Setting For each agent, we independently initialize each utility value by sampling from a normal distribution $\mathcal{N}(\mu_1, \sigma_1^2)$ with scalar mean $\mu_1 \in \mathbb{R}$ and variance $\sigma_1^2 \in \mathbb{R}$. We further normalize the utilities such that each agent's utility vector has a total sum of 1. In addition, we set each agent's excitement values by again independently sampling from a normal distribution $\mathcal{N}(\mu_2, \sigma_2^2)$. We clip each excitement value at 0 to ensure that the excitement is non-negative.

Algorithms We implement both the Men Proposing Deferred Acceptance (MPDA) and the Women Proposing Deferred Acceptance (WPDA) algorithms, which are guaranteed to return a stable matching. In addition, we implement a family of deterministic algorithms (Det) and a family of probabilistic algorithms (Prob), described as follows.

Det: This is a family of algorithms parameterized by $c \in [0, 1]$ and returns a perfect matching satisfying

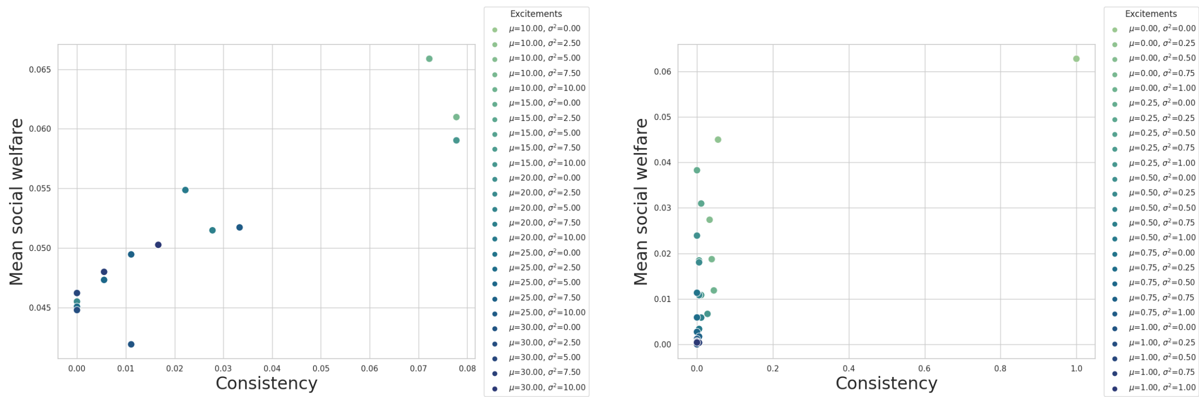
$$M^{t+1} = \underset{\text{consistency}(M^t, M) \geq c}{\arg\max} \text{sw}(M).$$

In other words, the generated matching M^{t+1} is the matching maximizing the social welfare subjected to the condition that the consistency is at least c compared to the most recent match. For computational efficiency, instead of enumerating through all possible matching that has at least consistency c, we approximate by fixing $N * c$ couples in M^t that have the highest social welfare according to the updated utilities $\overrightarrow{u}^{t+1}$, $\overrightarrow{v}^{t+1}$, and then perform the Hungarian Algorithm [6, 7, 8] over the remaining men and women to obtain the maximum weight perfect bipartite matching, with the weight between a man node m_i and a woman node w_j being $u_i^{t+1}(w_j) + v_j^{t+1}(m_i)$. As a result, the generated new matching has at least consistency c and a reasonably large social welfare.

4.2 Results

0.5 - 1 page Figures and analysis

- Evaluation of the algorithms (trade-off plot)
  - scatter plot for the algorithm family
  - variance plot for the algorithm family
- Evaluation with various dynamics
  - Plot the trade off curve with MPDA and update with match



(a) Dynamic Initialization (b) Dynamic Excitement

Figure 1: Mean social welfare vs. consistency for MPDA applied to two dyanmic scenarios; (a) fixed excitement ($\mu = 0.5, \sigma^2 = 0.5$) with varying initialization ($\mu \in [10, 20], \sigma^2 \in [0, 10]$); (b) fixed initialization ($\mu = 20, \sigma^2 = 5$) with varying excitement ($\mu \in [0, 1], \sigma^2 \in [0, 1]$).

5 Discussion

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